Notes for Math 451

Advanced Calculus I

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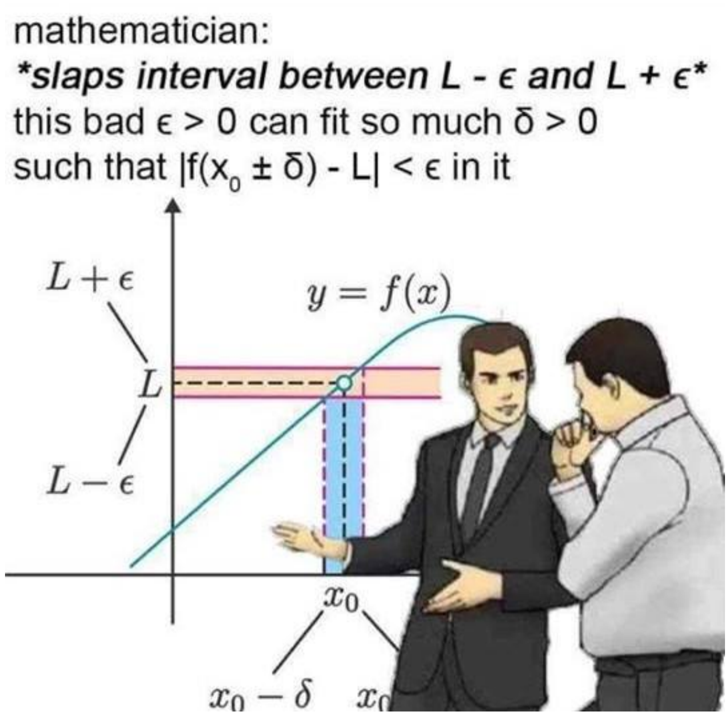
*Additional measure theory content.

0 Introduction

Course in elementary analysis. Sequences, differentiation, and integration, with additional notes on measure theory basics.

Textbook: *Elementary Analysis: The Theory of Calculus* by Ross. *Measures, Integrals and Martingales* by Schilling.



Source: "Measure 0 Memes for Lebesgue Integrable Teens"

1 Sets

1.1 The Natural Numbers $\mathbb{N}$

$\mathbb{N} = \{0, 1, 2, 3, \ldots\} \subset \mathbb{Z}$ is the set of natural numbers (some authors exclude 0). Below are some properties of $\mathbb{N}$:

N1) $\mathbb{N}$ is not empty.

N2) $\mathbb{N}$ has a smallest element.

N3) Every $n \in \mathbb{N}$ has a successor $n + 1 \in \mathbb{N}$.

N4) If $X \subset \mathbb{N}$ is such that $0 \in X$ and $n \in X \longrightarrow n + 1 \in X$, then $X = \mathbb{N}$.

The last property says that $\mathbb{N}$ is the smallest set with the first three properties. It is then natural to conjecture the following:

Claim. The four properties above uniquely characterize $\mathbb{N}$.

We will formulate this with more precise language later. First, we introduce some familiar technology.

Theorem 1.1.1 (Induction). *Let $P(n)$ be a logical statement with parameter $n \in \mathbb{N}$. Assume $P(0)$ is true and $P(n) \longrightarrow P(n+1)$. Then $\forall n \in \mathbb{N}$, $P(n)$ is true.*

Proof. Define $X := \{n \in \mathbb{N} \mid P(n) \text{ is true}\} \subset \mathbb{N}$. Then, since $0 \in X$ and $n \in X \longrightarrow n + 1 \in X$, we know $X = \mathbb{N}$. □

We also introduce the concept of recursion. To construct a collection $(S_n)_{n \in \mathbb{N}}$ of sets or maps, it is enough to construct S_0 and S_{n+1} given S_n. For instance, to construct $f : \mathbb{N} \longrightarrow S$, it is enough to specify $f(0) \in S$ and $f(n+1) \in S$ given $f(n) \in S$.

Lemma 1.1.1. Let $(S_n)_{n \in \mathbb{N}}$ and $(S'_n)_{n \in \mathbb{N}}$ be collections of sets. Assume $S_0 = S'_0$ and $S_n = S'_n \longrightarrow S_{n+1} = S'_{n+1}$. Then $S_n = S'_n$ for $n \in \mathbb{N}$.

Proof. Follows directly from induction on n. □

Definition 1.1.1. A **Peano triple** (P, e, s) consists of:

- a set P
- an element $e \in P$
- an injective map $s : P \longrightarrow P$

such that

P1) $e \notin S(P)$

P2) If $X \subset P$ is such that $e \in X$ and $S(X) \subset X$, then $X = P$.

Peano triples are essentially abstractions of the properties of $\mathbb{N}$ we stated above. For instance, it is easy to show that for a Peano triple (P, e, s), we have $P = \{e\} \cup S(P)$ using P2). If we use the successor function for s, the result mirrors property N4).

Now we can address our conjecture from earlier.

> **Theorem 1.1.2.** *Let (P, e, s) be a Peano triple. There exists a unique bijection $f : \mathbb{N} \longrightarrow P$ such that $f(0) = e$ and $f(n+1) = s(f(n))$.*

This result says that, for any Peano triple (P, e, s), we can map every natural number n to one element in P whose successor is the image of $n + 1$. That is, all Peano triples are equivalent to $\mathbb{N}$ up to bijections.

Proof. f is recursively defined, meaning it is unique by Lemma 1.1.1. It suffices to show it is bijective.

For injectivity, define the logical statement $T(n) := (\forall m \in \mathbb{N} : f(n) = f(m) \Longrightarrow n = m)$. We induce on n. First, consider $T(0)$ and take $m \in \mathbb{N}$. If $m = n = 0$, we are done. Otherwise, we can write $m = m' + 1$ for $m' \in \mathbb{N}$, and we write

$$f(m) = f(m' + 1) = S(f(m')) \in S(P).$$

By definition, $e \notin S(P)$, so $f(m) \neq e = f(n)$. This shows the contrapositive of $T(0)$.

Now suppose $T(n)$. Again, we will show the contrapositive. Take $m \in \mathbb{N}$ such that $m \neq n + 1$. If $m = 0$, we are done by $T(0)$. Otherwise, $m = m' + 1$ for $m' \in \mathbb{N}$. So $m' + 1 \neq n + 1 \Longrightarrow m' \neq n$, and $f(m') \neq f(n)$ by assumption. It follows that

$$f(n+1) = s(f(n)) \neq s(f(m')) = f(m' + 1)$$

since s is injective.

It remains to show surjectivity. Let $X = f(\mathbb{N}) \subseteq P$. Now $e = f(0) \in f(\mathbb{N}) = X$ and $s(X) = s(f(\mathbb{N})) = f(\mathbb{N} + 1) \subseteq f(\mathbb{N}) = X$. By P2), $X = P$, completing the proof. □

1.2 The Integers $\mathbb{Z}$

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is the set of integers. Although the construction of $\mathbb{Z}$ as "the natural numbers and their negatives" is intuitive, it would be nice to define $\mathbb{Z}$ in a way that only uses $\mathbb{N}$ and its axioms without resorting to ad-hoc definitions like "negative" and their behavior with arithmetic.

One construction might be to represent each integer as a difference of natural numbers. Since $-5 = 0 - 5$, we would represent -5 as $(0, 5)$. We would also need a new notion of equality since $(0, 5)$ and $(1, 6)$ represent the same number; perhaps $(a, b) \equiv (a', b') \Longleftrightarrow a - b = a' - b'$.

[to be continued...]

1.3 The Rational Numbers $\mathbb{Q}$

$\mathbb{Q} := \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$ is the set of rational numbers. $3 \in \mathbb{Q}$, while $\sqrt{2} \notin \mathbb{Q}$. The latter is part of a more general set called the algebraic numbers:

> **Definition 1.3.1.** An **algebraic number** is any r that satisfies
>
> $$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + x_0 = 0$$
>
> where each $c_i \in \mathbb{Z}$, $c_n \neq 0$, and $n \geq 1$.

Example 1.3.1. The quantity

$$\sqrt{\frac{4 - 2\sqrt{3}}{7}}$$

is an algebraic number.

Proof. Denote the given value by a. Then $a^2 = \frac{4-2\sqrt{3}}{7}$, so $2\sqrt{3} = 4 - 7a^2$, which expands to $49a^4 - 56a^2 + 4 = 0$. Therefore, a is a root of $49x^4 - 56x^2 + 4 = 0$. □

The following useful result is called the Rational Root Theorem.

Theorem 1.3.1. *Consider the polynomial equation*

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where each $c_i \in \mathbb{Z}$, $c_n \neq 0$, and $n \geq 1$. Let $r = \frac{c}{d}$ be a rational root, where c, d are coprime integers. Then $c \mid c_0$ and $d \mid c_n$.

Proof. We write

$$c_n \left(\frac{c}{d}\right)^n + c_{n-1}\left(\frac{c}{d}\right)^{n-1} + \cdots + c_1\left(\frac{c}{d}\right) + c_0 = 0.$$

Multiply both sides by d^n to obtain

$$c_n c^n + c_{n-1} c^{n-1} d + \cdots + c_1 c d^{n-1} + c_0 d^n = 0.$$

Therefore,

$$-c_n c^n = c_{n-1} c^{n-1} d + \cdots + c_1 c d^{n-1} + c_0 d^n.$$

Since d divides the right side, it must also divide $-c_n c^n$. But because $(c, d) = 1$, we also have $(c^n, d) = 1$, so $d \mid c_n$. We arrive at $c \mid c_0$ analogously after solving for $c_0 d^n$. □

The RRT is especially useful in the special case of $c_n = 1$. Then, since $d \mid c_n$, we have $d = 1$, so the only possible rational roots of a monic polynomial are integers. We can use this fact to determine whether certain numbers are rational.

Example 1.3.2. $\sqrt[3]{6} \notin \mathbb{Q}$.

Proof. The quantity is a solution of $x^3 - 6 = 0$. By RRT, the only possible rational solutions are $\pm 1, \pm 2, \pm 3, \pm 6$. By inspection, none of these are actually roots. Therefore, any solution to the equation, including the given quantity, is irrational. □

1.4 The Real Numbers $\mathbb{R}$

Definition 1.4.1. Take a set F. Then $(F, +, \cdot)$ is a **field** if:

1. $0 \neq 1$
2. $+$ and $\cdot$ are associative
3. $+$ and $\cdot$ are commutative

4. 0 is the additive identity
5. 1 is the multiplicative identity
6. + and $\cdot$ inverses exist
7. $\cdot$ distributes over +

It is an **ordered field** with order structure $\leq$ if, for $a, b, c \in F$:

1. $a \leq b$ or $b \leq a$
2. If $a \leq b$ and $b \leq a$, then $a = b$
3. If $a \leq b$ and $b \leq c$, then $a \leq c$
4. If $a \leq b$, then $a + c \leq b + c$
5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$

$\mathbb{Q}$ and $\mathbb{R}$ are ordered fields. These properties follow directly from the field axioms:

Theorem 1.4.1. *For a field F and $a, b, c \in F$:*

1. $a + c = b + c \implies a = b$
2. $a \cdot 0 = 0$
3. $a(-b) = -ab$
4. $(-a)(-b) = ab$
5. *If $c \neq 0$, then $ac = bc \implies a = b$*
6. *$ab = 0$ implies $a = 0$ or $b = 0$*

Proof.

1. Follows from right addition of $-c$ to both sides.
2. See 412 notes.
3. $ab + a(-b) = a(b + (-b)) = a \cdot 0 = 0$. So $a(-b)$ is the additive inverse of ab, as desired.
4. $(-a)(-b) + a(-b) = (a + (-a))(-b) = 0 \cdot (-b) = 0$. By the previous part, $(-a)(-b)$ then equals ab, the additive inverse of $-ab$.
5. Follows from right multiplication by c^{-1} on both sides.
6. Suppose $b \neq 0$ and $ab = 0$. Then $0 = ab(b^{-1}) = a$. Otherwise, done.

□

We can also prove some results using the ordered field axioms:

Theorem 1.4.2. *For a field F and $a, b, c \in F$:*

1. $a \leq b \implies -b \leq -a$
2. *$a \leq b$ and $c \leq 0$ implies $bc \leq ac$*
3. *$0 \leq a$ and $0 \leq b$ implies $0 \leq ab$*

4. $0 \le a^2$ *for all* a
5. $0 < 1$
6. $0 < a$ *implies* $0 < a^{-1}$
7. $0 < a < b$ *implies* $0 < b^{-1} < a^{-1}$

Proof.

1. $a \le b$ implies $a + (-a + (-b)) \le b + (-a + (-b))$, so $-b \le -a$.
2. By the previous part, $0 \le -c$, so $-ac \le -bc$ and $bc \le ac$.
3. $0 \cdot a \le ba \implies 0 \le ab$.
4. $0 \le a$ is straightforward. If $a \le 0$, we have $0 \le a \cdot a = a^2$ by (1).
5. Suppose $1 \le 0$. Then $0 \cdot 1 \le 1 \cdot 1 \implies 0 \le 1$, a contradiction.
6. Suppose $0 < a$ but $a^{-1} < 0$. Then $0 \cdot a^{-1} > aa^{-1} \implies 0 > 1$, a contradiction.
7. Adapt the proof of (1) using multiplicative inverses to obtain $b^{-1} < a^{-1}$. Then $0 < b^{-1}$ follows from (5).

□

Now we introduce absolute value and the concept of distance.

Definition 1.4.2. For $a \in \mathbb{R}$, the **absolute value** of a, denoted $|a|$, is the following function:

$$|x| = \begin{cases} x & x \ge 0 \\ -x & x \le 0 \end{cases}$$

Definition 1.4.3. For $a, b \in \mathbb{R}$, the **distance** between a and b, denoted $\text{dist}(a, b)$, is defined as $\text{dist}(a, b) = |a - b|$.

Theorem 1.4.3. *Take* $a, b \in \mathbb{R}$. *Then the following properties hold:*

1. $|a| \ge 0$
2. $|ab| = |a| \cdot |b|$
3. $|a + b| \le |a| + |b|$

Proof.

1. Follows by definition.
2. It is straightforward to check that if a and b have the same sign, $|ab| = |a| \cdot |b| = ab$. Otherwise, $|ab| = |a| \cdot |b| = -ab$.
3. By definition, $-|a| \le a \le |a|$ and $-|b| \le b \le |b|$. So $-|a| - |b| \le a + b \le |a| + |b|$. This implies $\pm(a + b) \le |a| + |b|$, so $|a + b| \le |a| + |b|$.

□

The last result is also called the **Triangle Inequality** because for $x, y, z \in \mathbb{R}$, we can substitute $a = x - y$ and $b = y - z$ to obtain $|x - z| \leq |x - y| + |y - z| \Rightarrow \text{dist}(x, z) \leq \text{dist}(x, y) + \text{dist}(y, z)$. Geometrically, this is analogous to the statement that the combined length of any two sides of a triangle is greater than the length of the third.

1.5 The Completeness Axiom

Some sets have "gaps." For instance, the graph of $x^2 - 2 = 0$ intersects with the x-axis twice—at $(\pm\sqrt{2}, 0)$. Both x-intercepts are irrational, so the parabola passes through two "gaps" in the rational numbers.

$\mathbb{R}$, on the other hand, is complete: it has no such gaps. This is provided by the **completeness axiom**. Firstly, we will introduce some terminology.

> **Definition 1.5.1.** Take a non-empty $S \subseteq \mathbb{R}$. If $s_0 \in S$ and $s \leq s_0$ for any $s \in S$, then s_0 is the **maximum** of S, denoted $s_0 = \max S$. We define the **minimum** $\min S$ analogously.

Every finite, non-empty subset of $\mathbb{R}$ has a maximum and minimum, but the same is not true for subsets like $(1, 3]$ or $\mathbb{Z}$.

> **Definition 1.5.2.** Take a non-empty $S \subseteq \mathbb{R}$. If there exists $M \in \mathbb{R}$ such that $s \leq M$ for all $s \in S$, then M is an **upper bound of** S and the set is **bounded above**. The **lower bound** is defined analogously; if one exists, S is **bounded below**.
>
> S is **bounded** if it is bounded above and below.

Note that a lower/upper bound does not need to be in the set, nor is it in general unique. Consider the set $A = \{r \in \mathbb{Q} \mid 0 \leq r \leq \sqrt{2}\}$. Any non-positive real number is a lower bound, and any real number at least $\sqrt{2}$ is an upper bound.

We can, however, say that 0 is the largest lower bound and $\sqrt{2}$ is the smallest upper bound. This motivates the next definition:

> **Definition 1.5.3.** Take a non-empty $S \subseteq \mathbb{R}$. If S is bounded above and has a least upper bound s_u, we say s_u is the **supremum** of S and write $s_u = \sup S$.
>
> The greatest lower bound s_l, if it exists, is called the **infimum** of S and is denoted $s_l = \inf S$.

So, from earlier, we have $\sup A = \sqrt{2}$ and $\inf A = 0$. In general, if a set has a maximum/minimum, it is also the set's supremum/infimum, respectively.

Some sets, like open intervals, do not have a min/max but do have a sup/inf. For instance, $B = \{x \in \mathbb{R} \mid x^2 < 10\} = (-\sqrt{10}, \sqrt{10})$ has $\sup B = \sqrt{10}$ and $\inf B = -\sqrt{10}$ but no min/max.

Now we introduce the **completeness axiom**:

> **Axiom 1.5.1.** Every non-empty $S \subseteq \mathbb{R}$ that is bounded above has a least upper bound. That is, $\sup S$ exists and is a real number.

This doesn't hold for $\mathbb{Q}$, and the set A from above is a counterexample since $\sqrt{2} \notin \mathbb{Q}$. We can show an analogous result for sets bounded below.

Corollary 1.5.1. *Every non-empty* $S \subseteq \mathbb{R}$ *that is bounded below has a greatest upper bound. That is,* $\inf S$ *exists and is a real number.*

Proof. Consider $-S = \{-s \mid s \in S\}$. We claim $-\sup(-S) = \inf S$. For intuition, plot each point of S on a number line. The leftmost bound corresponds to the rightmost one when S is reflected about 0. And since $-S \subseteq \mathbb{R}$, $\sup(-S)$ does in fact exist.

Denote $s_0 = \sup(-S)$. We must first show that $-s_0$ is a lower bound of S, namely $-s_0 \leq s$ for all $s \in S$. By definition, we have $-s \leq s_0$, and the result directly follows.

We also need to show $-s_0$ is the greatest lower bound: if $c \leq s$ for all $s \in S$, then $c \leq -s_0$. Let $d = -c$; then we have $-s \leq d$. But by construction of s_0, this implies $s_0 \leq d$, so $c = -d \leq -s_0$, as desired. □

Here are some more intuitive results about $\mathbb{Q}$ and $\mathbb{R}$.

Theorem 1.5.1 (Archimedian Property)**.** *If* $a > 0$ *and* $b > 0$*, then there exists positive* $n \in \mathbb{Z}$ *where* $na > b$*.*

Proof. Suppose the contrary; that there exist $a, b > 0$ where $na \leq b$ for all $n \in \mathbb{N}$. Then b is an upper bound for $S = \{na \mid n \in \mathbb{N}\}$. Since $S \subseteq \mathbb{R}$, $s_0 = \sup S$ exists.

Now note $s_0 - a < s_0$, so $s_0 - a$ is not an upper bound of S. That is, there exists $n_0 \in \mathbb{N}$ such that $s_0 - a < n_0 a$. But this implies $s_0 < (n_0 + 1)a \in S$, a contradiction. □

Theorem 1.5.2 (Denseness of $\mathbb{Q}$)**.** *If* $a, b \in \mathbb{R}$ *and* $a < b$*, there exists* $r \in \mathbb{Q}$ *such that* $a < r < b$*.*

Proof. If we write $r = \frac{m}{n}$ for $m, n \in \mathbb{Z}$, it suffices to show $an < m < bn$. Firstly, since $b - a > 0$, we pick $n \in \mathbb{N}$ such that $n(b - a) > 1 \implies bn - an > 1$, which exists due to the Archimedian property.

We now use the fact that the Archimedian property directly implies, for any real number, there exists a larger natural number. Let $k > \max(|an|, |bn|)$ where $k \in \mathbb{Z}$, so $-k < an < bn < k$. We then construct $K = \{j \in \mathbb{Z} \mid -k \leq j \leq k\}$ and $T = \{j \in K \mid an < j\}$, which are both non-empty since they both contain k. Denote $m = \min T$; then $-k < an < m$.

But since $m > -k$, we have $m - 1 \in K$. By choice of m, we know $m - 1 \notin T \implies m - 1 \leq an$, which implies $m \leq an + 1 < bn$.

Combining $an < m$ and $m < bn$, we obtain $an < m < bn$, as desired. □

Finally, we address the symbols $+\infty$ and $-\infty$. They are *not* real numbers, but they are useful in expressing unbounded intervals. We can write $[a, \infty) = \{k \in \mathbb{R} \mid k \geq a\}$.

Also, we write $\sup S = +\infty$ if S is not bounded above, and analogously for $\inf S = -\infty$.

2 Sequences

2.1 Limits of Sequences

Definition 2.1.1. A **sequence** is a function whose domain is a set of the form $\{n \in \mathbb{Z} \mid n \geq m\}$. Usually, $m \in \{1, 0\}$.

By convention, we denote the sequence by s and the value at n by s_n. The entire sequence can be written as $(s_n)_{n\in\mathbb{N}}$, or, more generally, $(s_n)_{n=m}^{\infty}$. Sometimes, we drop the subscript and write (s_n) when the value of m is understood from context or irrelevant.

For instance, $(s_n)_{n\in\mathbb{N}}$ where $s_n = \frac{1}{n^2}$ corresponds to the sequence $(1, \frac{1}{4}, \frac{1}{9}, \ldots)$.

Definition 2.1.2. A sequence of numbers **converges** to $s \in \mathbb{R}$ if, for all $\epsilon > 0$ there exists a number N such that $n > N$ implies $|s_n - s| < \epsilon$.

If (s_n) converges to s, we write $\lim_{n\to\infty} s_n = s$ or simply $s_n \longrightarrow s$, and s is called the **limit** of (s_n). If (s_n) does not converge to a real number, it diverges.

Example 2.1.1. Prove $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$.

Proof. We write

$$\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < \epsilon \iff \left|\frac{19}{7(7n-4)}\right| < \epsilon \iff \frac{19}{49\epsilon} + \frac{4}{7} < n.$$

The last step follows because $7(7n - 4) > 0$ if we pick a positive n.

So, for all $\epsilon > 0$, we have $n > \frac{19}{49\epsilon} + \frac{4}{7}$ implies $\frac{3n+1}{7n-4} - \frac{3}{7} < \epsilon$. □

3 σ-algebras*

3.1 The Basics

Given a set X and $A \subset X$, denote $A^c = X \backslash A$.

Definition 3.1.1. A **σ-algebra** $\mathcal{A}$ on a set X is a family of subsets of X with the following properties:

1. $X \in \mathcal{A}$
2. $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
3. $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A} \Rightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$

A set $A \in \mathcal{A}$ is said to be **measurable** or **$\mathcal{A}$-measurable**.

The third requirement says that the union of countably many subsets of $\mathcal{A}$ must also be a subset of $\mathcal{A}$.

Theorem 3.1.1. *Consider a σ-algebra $\mathcal{A}$. Then the following properties hold:*

- $\emptyset \in \mathcal{A}$
- $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$
- $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A} \Rightarrow \bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

Proof.

- Since $X \in \mathcal{A}$, we write $\emptyset = X^c \in \mathcal{A}$ by properties 1 and 2.
- Follows directly from property 3.
- By property 2, we write $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A} \Rightarrow (A_n^c)_{n\in\mathbb{N}} \subset \mathcal{A}$. Then we use DeMorgan to obtain

$$\left(\bigcap_{n\in\mathbb{N}} A_n\right)^c = \bigcup_{n\in\mathbb{N}} A_n^c \in \mathcal{A} \Rightarrow \bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}.$$

□

Example 3.1.1. Denote the cardinality of a set A by $\#A$. Show that the following set is a σ-algebra:

$$\mathcal{A} := \{A \subset X : \#A \leq \mathbb{N} \text{ or } \#A^c \leq \mathbb{N}\}.$$

That is, $\mathcal{A}$ is the set of countable subsets of X and their complements.

Proof. We show that $\mathcal{A}$ satisfies the three properties of a σ-algebra.

1. $X^c = \emptyset$, which is countable. Hence $X \in \mathcal{A}$.
2. If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$ because $(A^c)^c = A$.

3. Fix a set of (A_n) and suppose all are countable. Then $\bigcup_{n\in\mathbb{N}} A_n$ is the countable union of countable sets; hence, is is countable.

   Now suppose some $A_i \in \mathcal{A}$ is uncountable. Then A_i^c must be countable, so we write

$$\left(\bigcup_{n\in\mathbb{N}} A_n\right)^c = \bigcap_{n\in\mathbb{N}} A_n^c \subset A_i^c.$$

   So the leftmost expression is countable, and thus its complement is in $\mathcal{A}$.

□

Theorem 3.1.2 (Existence of generators)**.** *For every system of sets $\mathcal{G} \subset \mathcal{P}(X)$ there exists a smallest σ-algebra containing $\mathcal{G}$.*

Proof. Consider the union of all σ-algebras containing $\mathcal{G}$:

$$\mathcal{A} := \bigcap_{\mathcal{F}\supset\mathcal{G}} \mathcal{F}, \quad \text{where } \mathcal{F} \text{ is a } \sigma\text{-algebra.}$$

We claim that $\mathcal{A}$ is the minimal family in question. Using Definition 3.1.1, it is easy to check that the intersection of arbitrarily many σ-algebras is itself a σ-algebra.

But, by definition, if $\mathcal{G} \subset \mathcal{A}'$ for a σ-algebra $\mathcal{A}'$, then $\mathcal{A} \subset \mathcal{A}'$, so $|\mathcal{A}| \leq |\mathcal{A}'|$. So $\mathcal{A}$ is the smallest σ-algebra containing $\mathcal{G}$. □

3.2 Borel σ-algebras